Approximation in the Zygmund Class

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A Short Motivation

Consider the spaces of functions $f\colon I_0 = [0,1] \to \mathbb{R}$

- L^p, for $1 \leqslant p < \infty$, with

$$\|f\|_{L^p} = \left(\int_{I_0} |f(t)|^p \, dt\right)^{1/p},$$

- L^∞, with

$$\|f\|_{L^\infty} = \sup_{t \in I_0} |f(t)|,$$

and

- BMO, with

$$\|f\|_{\mathrm{BMO}} = \sup_{I \subseteq I_0} \left(\frac{1}{|I|} \int_I |f(t) - f_I|^2 \, dt\right)^{1/2},$$

where $f_I = \int_I f(t)\, dt$

It is known that

$$L^\infty \subsetneq \mathrm{BMO} \subsetneq L^p \subsetneq L^q \subsetneq L^1, \quad \text{for } 1 < p < q < \infty$$

A singular integral operator (e.g. H the Hilbert Transform) is bounded

- from L^p to L^p if $1 < p < \infty$,
- from BMO to BMO and
- from L^∞ to BMO

J. Garnett and P. Jones (1978) characterised $\overline{L^\infty}$ for $\|\cdot\|_{\mathrm{BMO}}$

A Different Setting

Consider the spaces of continuous functions $f: I_0 \to \mathbb{R}$

- Lip_α, for $0 < \alpha \leqslant 1$, with

$$|f(x) - f(y)| \leqslant C|x - y|^\alpha, \quad x, y \in \mathbb{R},$$

and

- the *Zygmund class* Λ_*, with

$$\|f\|_* = \sup_{\substack{x \in I_0 \\ h > 0}} \frac{|f(x+h) - 2f(x) + f(x-h)|}{h} < \infty$$

It can be seen that

$$\mathrm{Lip}_1 \subsetneq \Lambda_* \subsetneq \mathrm{Lip}_\alpha \subsetneq \mathrm{Lip}_\beta, \qquad \text{for } 0 < \alpha < \beta < 1$$

Singular integral operators are bounded

- from Lip_α to Lip_α for $0 < \alpha < 1$,
- from Λ_* to Λ_* and
- from Lip_1 to Λ_*

What could be a characterisation of $\overline{\mathrm{Lip}_1}$ for $\|\cdot\|_*$?

Related open problem (J. Anderson, J. Clunie, C. Pommerenke; 1974):

- what is the characterisation of $\overline{\mathbb{H}^\infty}$ for the Bloch space norm?

Our Concepts

A function $f : I_0 \to \mathbb{R}$ belongs to

- the Zygmund class Λ_* if it is continuous and

$$\|f\|_* = \sup_{\substack{x \in I_0 \\ h>0}} \frac{|f(x+h) - 2f(x) + f(x-h)|}{h} < \infty,$$

- BMO if it is locally integrable and

$$\|f\|_{\text{BMO}} = \sup_{I \subseteq I_0} \left(\frac{1}{|I|} \int_I |f(t) - f_I|^2 \, dt \right) < \infty,$$

- I(BMO) if it is continuous and $f' \in \text{BMO}$ (distributional)

Note that $\text{I(BMO)} \subsetneq \Lambda_*$

Our Concepts

What is the characterisation of $\overline{\mathrm{I}(\mathrm{BMO})}$ for $\|\cdot\|_*$?

Related problem:
P. G. Ghatage and D. Zheng (1993) characterised $\overline{\mathrm{BMOA}}$ for the Bloch space norm

Notation

- Given $x \in \mathbb{R}$, $h > 0$, denote

$$\Delta_2 f(x,h) = \frac{f(x+h) - 2f(x) + f(x-h)}{h}$$

- If $I = (x-h, x+h)$, we say

$$\Delta_2 f(I) = \Delta_2 f(x,h)$$

- Given $f, g \in \Lambda_*$, consider

$$\mathrm{dist}(f,g) = \|f-g\|_*,$$

and given $X \subseteq \Lambda_*$, we say

$$\mathrm{dist}(f,X) = \inf_{g \in X} \|f-g\|_*$$

A Characterisation for I(BMO)

Theorem (R. Strichartz; 1980)

A continuous function f *is in* I(BMO) *if and only if*

$$\sup_{I \subseteq I_0} \left(\frac{1}{|I|} \int_I \int_0^{|I|} |\Delta_2 f(x,h)|^2 \frac{dh\, dx}{h} \right)^{1/2} < \infty$$

The Main Result

Given $\varepsilon > 0$ and $f \in \Lambda_*$, consider

$$A(f,\varepsilon) = \{(x,h) \in \mathbb{R} \times \mathbb{R}_+ \colon |\Delta_2 f(x,h)| > \varepsilon\}$$

Theorem

Let $f \in \Lambda_*$ be compactly supported on I_0. For each $\varepsilon > 0$ consider

$$C(f,\varepsilon) = \sup_{I \subseteq I_0} \frac{1}{|I|} \int_I \int_0^{|I|} \chi_{A(f,\varepsilon)}(x,h) \, \frac{dh\, dx}{h}.$$

Then,

$$\operatorname{dist}(f, \mathrm{I}(\mathrm{BMO})) \simeq \inf\{\varepsilon > 0 \colon C(f,\varepsilon) < \infty\}. \tag{1}$$

Denote by ε_0 the infimum in (1)

Generalisation to Zygmund Measures

- A measure μ on $\mathbb{R}^d$ is a Zygmund measure, $\mu \in \Lambda_*(\mathbb{R}^d)$, if

$$\|\mu\|_* = \sup_Q \left| \frac{\mu(Q)}{|Q|} - \frac{\mu(Q^*)}{|Q^*|} \right| < \infty$$

- A measure ν on $\mathbb{R}^d$ is I(BMO) if it is absolutely continuous and $d\nu = b(x)\,dx$, for some $b \in \text{BMO}$
- For $(x,h) \in \mathbb{R}^d \times \mathbb{R}_+$, let $Q(x,h)$ be a cube with centre x and $l(Q) = h$, and denote

$$\Delta_2\mu(x,h) = \frac{\mu(Q(x,h))}{|Q(x,h)|} - \frac{\mu(Q(x,2h))}{|Q(x,2h)|}$$

- Given $\varepsilon > 0$ and $\mu \in \Lambda_*$, consider

$$A(\mu,\varepsilon) = \{(x,h) \in \mathbb{R}^d \times \mathbb{R}_+ : |\Delta_2\mu(x,h)| > \varepsilon\}$$

Generalisation to Zygmund Measures

Theorem

Let $\mu \in \mathbb{R}^d$ be compactly supported on Q_0. For each $\varepsilon > 0$ consider

$$C(\mu, \varepsilon) = \sup_{Q \subseteq Q_0} \frac{1}{|Q|} \int_Q \int_0^{l(Q)} \chi_{A(\mu,\varepsilon)}(x, h) \, \frac{dh \, dx}{h}.$$

Then,

$$\operatorname{dist}(\mu, \mathrm{I}(\mathrm{BMO})) \simeq \inf\{\varepsilon > 0 \colon C(\mu, \varepsilon) < \infty\}.$$

Further Results and Open Problem

- Generalisation for Zygmund measure μ on $\mathbb{R}^d$, $d \geqslant 1$ that is for μ with
$$\|\mu\|_* = \sup_Q \left| \frac{\mu(Q)}{|Q|} - \frac{\mu(Q^*)}{|Q^*|} \right| < \infty$$

- Application to functions in the Zygmund class that are also $W^{1,p}$, for $1 < p < \infty$
- We can't generalise the results for functions on $\mathbb{R}^d$ for $d > 1$

The Idea for the Proof

$$A(f,\varepsilon) = \{(x,h) \in \mathbb{R} \times \mathbb{R}_+ : |\Delta_2 f(x,h)| > \varepsilon\}$$

Theorem

Let $f \in \Lambda_*$. For each $\varepsilon > 0$ consider

$$C(f,\varepsilon) = \sup_{I \subseteq I_0} \frac{1}{|I|} \int_I \int_0^{|I|} \chi_{A(f,\varepsilon)}(x,h)\, \frac{dh\,dx}{h}.$$

Then,

$$\operatorname{dist}(f, \mathrm{I(BMO)}) \simeq \varepsilon_0 = \inf\{\varepsilon > 0 \colon C(f,\varepsilon) < \infty\}.$$

The easy part is to show that $\operatorname{dist}(f, \mathrm{I(BMO)}) \geqslant \varepsilon_0$

Our Tools

- I is dyadic if it is $I = [k2^{-n}, (k+1)2^{-n})$, with $n \geqslant 0$ and $0 \leqslant k < 2^n - 1$
- $\mathcal{D}$ the set of dyadic intervals and $\mathcal{D}_n$ the set of dyadic intervals of size 2^{-n}
- $S = \{S_n\}_{n \geqslant 0}$ is a dyadic martingale if
    - $S_n = S_n(I)$ constant on any $I \in \mathcal{D}_n$ for $n \geqslant 0$ and
    - $S_n(I) = \frac{1}{2}(S_{n+1}(I_+) + S_{n+1}(I_-))$
- $\Delta S(I) = S_n(I) - S_{n-1}(I^*)$, for $I \in \mathcal{D}_n$ and $I \subseteq I^* \in \mathcal{D}_{n-1}$
- For f continuous, define the average growth martingale S by

$$S_n(I) = \frac{f(b) - f(a)}{2^{-n}}, \quad I \in \mathcal{D}_n,$$

for each $n \geqslant 1$ and where $I = (a, b)$

- if $f \in \Lambda_*$, its average growth martingale S satisfies

$$\|S\|_* = \sup_{I \in \mathcal{D}} |\Delta S(I)| < \infty$$

- if $b \in \mathrm{BMO}$, its average growth martingale B satisfies

$$\|B\|_{\mathrm{BMO}} = \sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} |\Delta B(J)|^2 |J| \right)^{1/2} < \infty$$

These are related to the dyadic versions of our spaces...

A function $f\colon I_0 \to \mathbb{R}$ belongs to

- the *dyadic* Zygmund class Λ_{*d} if it is continuous and

$$\|f\|_{*d} = \sup_{I \in \mathcal{D}} |\Delta_2 f(I)| < \infty,$$

- to BMO_d if it is locally integrable and

$$\|f\|_{\mathrm{BMO}\,d} = \sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \int_I |f(t) - f_I|^2 \, dt \right)^{1/2} < \infty,$$

- to $\mathrm{I(BMO)}_d$ if it is continuous and $f' \in \mathrm{BMO}_d$ (distributional)

The Easy Version of the Theorem

Theorem

Let $f \in \Lambda_{*d}$. *For each* $\varepsilon > 0$ *consider*

$$D(f,\varepsilon) = \sup_{I \in \mathcal{D}} \frac{1}{|I|} \sum_{\substack{J \in \mathcal{D}(I) \\ |\Delta_2 f(J)| > \varepsilon}} |J|.$$

Then,

$$\operatorname{dist}(f, \mathrm{I}(\mathrm{BMO})_d) = \inf\{\varepsilon > 0 \colon D(f,\varepsilon) < \infty\}.$$

We can construct a function that approximates f using martingales

The Last Pieces

Theorem (Garnett-Jones, 1982)

Let $\alpha \mapsto b^{(\alpha)}$ *be measurable from* $\mathbb{R}$ *to* BMO_d*, all* $b^{(\alpha)}$ *supported on a compact* I_0*, with* $\left\| b^{(\alpha)} \right\|_{\mathrm{BMO}\,d} \leqslant 1$ *and*

$$\int b^{(\alpha)}(t)\, dt = 0.$$

Then,

$$b_R(t) = \frac{1}{2R} \int_{-R}^{R} b^{(\alpha)}(t+\alpha)\, d\alpha$$

is in BMO *and there is* $C > 0$ *such that* $\|b_R\|_{\mathrm{BMO}} \leqslant C$ *for any* $R > 0$.

The Last Pieces

Theorem

Let $\alpha \mapsto h^{(\alpha)}$ *be measurable from* $\mathbb{R}$ *to* Λ_{*d}*, all* $h^{(\alpha)}$ *supported on a compact* I_0*, with* $\left\|h^{(\alpha)}\right\|_{*d} \leqslant 1$*. Then,*

$$h_R(t) = \frac{1}{2R}\int_{-R}^{R} h^{(\alpha)}(t+\alpha)\,d\alpha$$

is in Λ_* *and there is* $C > 0$ *such that* $\|h_R\|_* \leqslant C$ *for any* $R > 0$*.*

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